

Growth of sumsets in abelian semigroups*

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Let S be an abelian semigroup, written additively, that contains the identity element 0. Let A be a nonempty subset of S . The cardinality of A is denoted $|A|$. For any positive integer h , the *sumset* hA is the set of all sums of h not necessarily distinct elements of A . We define $hA = \{0\}$ if $h = 0$. Let A_1, \dots, A_r , and B be nonempty subsets of S , and let h_1, \dots, h_r be nonnegative integers. We denote by

$$B + h_1A_1 + \dots + h_rA_r \quad (1)$$

the set of all elements of S that can be represented in the form $b + u_1 + \dots + u_r$, where $b \in B$ and $u_i \in h_iA_i$ for all $i = 1, \dots, r$. If the sets A_1, \dots, A_r , and B are finite, then the sumset (1) is finite for all h_1, \dots, h_r . The *growth function* of this sumset is

$$\gamma(h_1, \dots, h_r) = |B + h_1A_1 + \dots + h_rA_r|.$$

For example, let S be the additive semigroup of nonnegative integers \mathbf{N}_0 , and let A_1, \dots, A_r , and B be nonempty, finite subsets of \mathbf{N}_0 , normalized so that $0 \in B \cap A_1 \cap \dots \cap A_r$ and $\gcd(A_1 \cup \dots \cup A_r) = 1$. Let $b^* = \max(B)$ and $a_i^* = \max A_i$ for $i = 1, \dots, r$. Han, Kirfel, and Nathanson [1, 5] determined the asymptotic structure of the sumset $B + h_1A_1 + \dots + h_rA_r$. They proved that there exist integers c and d and finite sets $C \subseteq [0, c-2]$ and $D \subseteq [0, d-2]$ such that

$$B + h_1A_1 + \dots + h_rA_r = C \cup [c, b^* + \sum_{i=1}^r a_i^* h_i - d] \cup \left(b^* + \sum_{i=1}^r a_i^* h_i - D \right).$$

for $\min(h_1, \dots, h_r)$ sufficiently large. This implies that the growth function is eventually a multilinear function of h_1, \dots, h_r , that is, there exists an integer Δ such that

$$|B + h_1A_1 + \dots + h_rA_r| = a_1^* h_1 + \dots + a_r^* h_r + b^* + 1 - \Delta$$

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for $\min(h_1, \dots, h_r)$ sufficiently large. The explicit determination of the sets C and D is a difficult unsolved problem in additive number theory. In the case $r = 1$, it is called the *linear diophantine problem of Frobenius*. For a survey of finite sumsets in additive number theory, see Nathanson [6].

The theorem about sums of finite sets of integers generalizes to sums in an arbitrary abelian semigroup S . We shall prove that if A_1, \dots, A_r , and B are finite, nonempty subsets of S , then the growth function $\gamma(h_1, \dots, h_r)$ is eventually polynomial, that is, there exists a polynomial $p(z_1, \dots, z_r)$ such that

$$\gamma(h_1, \dots, h_r) = |B + h_1 A_1 + \dots + h_r A_r| = p(h_1, \dots, h_r)$$

for $\min(h_1, \dots, h_r)$ sufficiently large. The case $r = 1$ is due to Khovanskii [3, 4]. We use his method to extend the result to the case $r \geq 2$. The idea of the proof is to show that the growth function is the Hilbert function of a suitably constructed module graded by the additive semigroup \mathbf{N}_0^r of r -tuples of nonnegative integers.

We need the following result about Hilbert functions. Let R be a finitely generated \mathbf{N}_0^r -graded connected commutative algebra over a field E . Then $R = \bigoplus_{h \in \mathbf{N}_0^r} R_h$. Suppose that R is generated by s homogeneous elements y_1, \dots, y_s with $y_i \in R_{\delta_i}$, that is, the degree of y_i is $\deg y_i = \delta_i \in \mathbf{N}_0^r$. Let M be a finitely generated \mathbf{N}_0^r -graded R -module. For $h = (h_1, \dots, h_r) \in \mathbf{N}_0^r$, we define the Hilbert function

$$H(M, h) = \dim_E (M_{(h_1, \dots, h_r)}).$$

For $z = (z_1, \dots, z_r)$, we define

$$z^h = z_1^{h_1} \dots z_r^{h_r}.$$

Consider the formal power series

$$F(M, z) = \sum_{h \in \mathbf{N}_0^r} H(M, h) z^h.$$

Then there exists a vector β with integer coordinates and a polynomial $P(M, z) = P(M, z_1, \dots, z_h)$ with integer coefficients such that

$$F(M, z) = \frac{z^\beta P(M, z)}{\prod_{i=1}^s (1 - z^{\delta_i})}.$$

(This is Theorem 2.3 in Stanley [7, p. 33]).

Theorem 1 *Let A_1, \dots, A_r , and B be finite, nonempty subsets of an abelian semigroup S . There exists a polynomial $p(z_1, \dots, z_r)$ such that*

$$|B + h_1 A_1 + \dots + h_r A_r| = p(h_1, \dots, h_r)$$

for all sufficiently large integers h_1, \dots, h_r .

Proof. For $i = 1, \dots, r$, let

$$A_i = \{a_{i,1}, \dots, a_{i,k_i}\},$$

where

$$|A_i| = k_i \geq 1.$$

We introduce a variable $x_{i,j}$ for each $i = 1, \dots, r$ and $j = 1, \dots, k_i$. Fix a field E . We begin with the polynomial ring

$$R = E[x_{1,1}, \dots, x_{r,k_r}]$$

in the $s = k_1 + \dots + k_r$ variables $x_{i,j}$. The algebra R is connected since it is an integral domain (cf. Hartshorne [2, Exercise 2.19, p. 82]). For each r -tuple $(h_1, \dots, h_r) \in \mathbf{N}_0^r$ we let

$$R_{(h_1, \dots, h_r)}$$

be the vector subspace of R consisting of all polynomials that are homogeneous of degree h_i in the variables $x_{i,1}, \dots, x_{i,k_i}$. In particular, $E = R_{(0, \dots, 0)}$. Then

$$R = \bigoplus_{(h_1, \dots, h_r) \in \mathbf{N}_0^r} R_{(h_1, \dots, h_r)}.$$

The multiplication in the algebra R is consistent with this direct sum decomposition in the sense that

$$R_{(h_1, \dots, h_r)} R_{(h'_1, \dots, h'_r)} \subseteq R_{(h_1+h'_1, \dots, h_r+h'_r)},$$

and so R is graded by the semigroup \mathbf{N}_0^r .

Next we construct an \mathbf{N}_0^r -graded R -module M . To each r -tuple $(h_1, \dots, h_r) \in \mathbf{N}_0^r$ we associate a finite-dimensional vector space $M_{(h_1, \dots, h_r)}$ over the field E in the following way. To each element

$$u \in B + h_1 A_1 + \dots + h_r A_r$$

we assign the symbol

$$[u, h_1, \dots, h_r].$$

Let $M_{(h_1, \dots, h_r)}$ be the vector space consisting of all E -linear combinations of these symbols. Then

$$\dim_E M_{(h_1, \dots, h_r)} = |B + h_1 A_1 + \dots + h_r A_r|. \quad (2)$$

Let

$$M = \bigoplus_{(h_1, \dots, h_r) \in \mathbf{N}_0^r} M_{(h_1, \dots, h_r)}.$$

This is an \mathbf{N}_0^r -graded vector space over E .

To make M a module over the algebra R , we must construct a bilinear multiplication $R \times M \rightarrow M$. We define the product of the variable $x_{i,j} \in R$ and the basis element $[u, h_1, \dots, h_r] \in M$ as follows:

$$x_{i,j}[u, h_1, \dots, h_r] = [u + a_{i,j}, h_1, \dots, h_{i-1}, h_i + 1, h_{i+1}, \dots, h_r].$$

This makes sense since

$$u \in B + h_1 A_1 + \dots + h_i A_i + \dots + h_r A_r$$

and so

$$u + a_{i,j} \in B + h_1 A_1 + \dots + (h_i + 1) A_i + \dots + h_r A_r.$$

This induces a well-defined multiplication of elements of M by polynomials in R since, if $i < i'$,

$$\begin{aligned} x_{i',j'}(x_{i,j}[u, h_1, \dots, h_r]) &= x_{i',j'}[u + a_{i,j}, h_1, \dots, h_i + 1, \dots, h_r] \\ &= [u + a_{i,j} + a_{i',j'}, h_1, \dots, h_i + 1, \dots, h_{i'} + 1, \dots, h_r] \\ &= [u + a_{i',j'} + a_{i,j}, h_1, \dots, h_i + 1, \dots, h_{i'} + 1, \dots, h_r] \\ &= x_{i,j}[u + a_{i',j'}, h_1, \dots, h_{i'} + 1, \dots, h_r] \\ &= x_{i,j}(x_{i',j'}[u, h_1, \dots, h_r]). \end{aligned}$$

The case $i \geq i'$ is similar. Note that this is the only place where we use the commutativity of the semigroup S . It follows that M is an R -module. Moreover,

$$R_{(h_1, \dots, h_r)} M_{(h'_1, \dots, h'_r)} \subseteq M_{(h_1 + h'_1, \dots, h_r + h'_r)},$$

and so M is a graded R -module. Furthermore, the finite set

$$\{[b, 0, \dots, 0] : b \in B\} \subseteq M$$

generates M as an R -module.

Since $x_{i,j} \in R_{\delta_{i,j}}$, where $\deg(x_{i,j}) = \delta_{i,j}$ is the r -tuple whose i -th coordinate is 1 and whose other coordinates are 0, and since

$$\frac{1}{(1 - z_i)^{k_i}} = \sum_{h_i=0}^{\infty} \binom{h_i + k_i - 1}{k_i - 1} z_i^{h_i},$$

we have

$$\begin{aligned} F(M, z) &= \sum_{h \in \mathbf{N}_0^r} H(M, h) z^h \\ &= \frac{z^\beta P(M, z)}{\prod_{i=1}^r \prod_{j=1}^{k_i} (1 - z^{\delta_{i,j}})} \\ &= \frac{z^\beta P(M, z)}{\prod_{i=1}^r (1 - z_i)^{k_i}} \end{aligned}$$

$$\begin{aligned}
&= z^\beta P(M, z) \prod_{i=1}^r \sum_{h_i=0}^{\infty} \binom{h_i + k_i - 1}{k_i - 1} z_i^{h_i} \\
&= z^\beta P(M, z) \sum_{h=(h_1, \dots, h_r) \in \mathbf{N}_0^r} \prod_{i=1}^r \binom{h_i + k_i - 1}{k_i - 1} z^h.
\end{aligned}$$

This implies that the Hilbert function $H(M, h)$ is a polynomial in h_1, \dots, h_r for $\min(h_1, \dots, h_r)$ sufficiently large. By (2), the growth function is the Hilbert function of M . This completes the proof.

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